Conic Sections: An Overview

Conic sections are the curves which can be derived from taking slices of a "double-napped" cone. (A double-napped cone, in regular English, is two cones "nose to nose", with the one cone balanced perfectly on the other.) "Section" here is used in a sense similar to that in medicine or science, where a sample (from a biopsy, for instance) is frozen or suffused with a hardening resin, and then extremely thin slices ("sections") are shaved off for viewing under a microscope. If you think of the double-napped cones as being hollow, the curves we refer to as conic sections are what results when you section the cones at various angles.

There are some basic terms that you should know for this topic:

- **center**: the point \((h, k)\) at the center of a circle, an ellipse, or an hyperbola.
- **vertex** (VUR-teks): in the case of a parabola, the point \((h, k)\) at the "end" of a parabola; in the case of an ellipse, an end of the major axis; in the case of an hyperbola, the turning point of a branch of an hyperbola; the plural form is "vertices" (VUR-tuh-seez).
- **focus** (FOH-kuss): a point from which distances are measured in forming a conic; a point at which these distance-lines converge, or "focus"; the plural form is "foci" (FOH-siy).
- **directrix** (dih-RECK-triks): a line from which distances are measured in forming a conic; the plural form is "directrices" (dih-RECK-trih-seez).
- **axis** (AK-siss): a line perpendicular to the directrix passing through the vertex of a parabola; also called the "axis of symmetry"; the plural form is "axes" (ACK-seez).
- **major axis**: a line segment perpendicular to the directrix of an ellipse and passing through the foci; the line segment terminates on the ellipse at either end; also called the "principal axis of symmetry"; the half of the major axis between the center and the vertex is the semi-major axis.
- **minor axis**: a line segment perpendicular to and bisecting the major axis of an ellipse; the segment terminates on the ellipse at either end; the half of the minor axis between the center and the ellipse is the semi-minor axis.
- **locus** (LOH-kuss): a set of points satisfying some condition or set of conditions; each of the conics is a locus of points that obeys some sort of rule or rules; the plural form is "loci" (LOH-siy).

Each conic has a "typical" equation form, sometimes along the lines of the following:

- **parabola**: \(Ax^2 + Dx + Ey = 0\)
- **circle**: \(x^2 + y^2 + Dx + Ey + F = 0\)
- **ellipse**: \(Ax^2 + Cy^2 + Dx + Ey + F = 0\)
- **hyperbola**: \(Ax^2 - Cy^2 + Dx + Ey + F = 0\)

You may have noticed, in the table of "typical" shapes (above), that the graphs either paralleled the \(x\)-axis or the \(y\)-axis, and you may have wondered whether conics can ever be "slanted", such as:

Yes, conic graphs can be "slanty", as shown above. But the equations for the "slanty" conics get so much more messy that you can't deal with them until after trigonometry. If you wondered why the coefficients in the "general conic" equations, such as \(Ax^2 + Cy^2 + Dx + Ey + F = 0\), skipped the letter \(B\), it's because the \(B\) is the coefficient of the "\(xy\)" term that you can't handle until after you have some trigonometry under your belt. Using parametric equations can make rotating conic sections a little easier.
PARABOLAS
The "vertex" form of a parabola with its vertex at \((h, k)\) is:
- regular: \(y = a(x - h)^2 + k\)
- sideways: \(x = a(y - k)^2 + h\)

The conics form of the parabola equation (the one you'll find in advanced or older texts) is:
- regular: \(4p(y - k) = (x - h)^2\)
- sideways: \(4p(x - h) = (y - k)^2\)

\(p\) represents the distance between the vertex and the focus, and also the same (that is, equal) distance between the vertex and the directrix. And \(2p\) is then clearly the distance between the focus and the directrix.

CIRCLES
In algebraic terms, a circle is the set (or "locus") of points \((x, y)\) at some fixed distance \(r\) from some fixed point \((h, k)\). The value of \(r\) is called the "radius" of the circle, and the point \((h, k)\) is called the "center" of the circle.

The "general" equation of a circle is:
\[x^2 + y^2 + Dx + Ey + F = 0\]

The "center-radius" form of the equation is:
\[(x - h)^2 + (y - k)^2 = r^2\]

ELLIPSES
For a wider-than-tall ellipse with center at \((h, k)\), having vertices \(a\) units to either side of the center and foci \(c\) units to either side of the center, the ellipse equation is:
\[\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1\]

For a taller-than-wide ellipse with center at \((h, k)\), having vertices \(a\) units above and below the center and foci \(c\) units above and below the center, the ellipse equation is:
\[\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1\]

An ellipse equation, in conics form, is always "=1". Note that, in both equations above, the \(h\) always stayed with the \(x\) and the \(k\) always stayed with the \(y\). The only thing that changed between the two equations was the placement of the \(a^2\) and the \(b^2\). The \(a^2\) always goes with the variable whose axis parallels the wider direction of the ellipse; the \(b^2\) always goes with the variable whose axis parallels the narrower direction. Looking at the equations the other way, the larger denominator always gives you the value of \(a^2\), the smaller denominator always gives you the value of \(b^2\), and the two denominators together allow you to find the value of \(c^2\) and the orientation of the ellipse.

HYPERBOLAS
When the transverse axis is horizontal (in other words, when the center, foci, and vertices line up side by side, parallel to the \(x\)-axis), then the \(a^2\) goes with the \(x\) part of the hyperbola's equation, and the \(y\) part is subtracted.
\[\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1\]

When the transverse axis is vertical (in other words, when the center, foci, and vertices line up above and below each other, parallel to the \(y\)-axis), then the \(a^2\) goes with the \(y\) part of the hyperbola's equation, and the \(x\) part is subtracted.
\[\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1\]

In "conics" form, an hyperbola's equation is always "=1".